

HOW EINSTEIN MIGHT HAVE BEEN LED TO RELATIVITY ALREADY IN 1895

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Point of departure: moving observer's description of an elastic wave. We—who call ourselves O to emphasize our inertiality—contemplate a taut string at rest. Having been recently introduced to the physics of waves, we remark that *elastic longitudinal vibrations* of the string are described by the wave equation

$$\left[\left(\frac{\partial}{\partial x} \right)^2 - \frac{1}{u^2} \left(\frac{\partial}{\partial t} \right)^2 \right] \phi(x, t) = 0 \quad (1)$$

and that the general solution of (1) can be represented

$$\begin{aligned} \phi(x, t) &= f(x - ut) + g(x + ut) \\ &= \text{rigidly right-running} + \text{rigidly left-running} \end{aligned} \quad (2)$$

where the wave function $\phi(x, t)$ refers physically to the *instantaneous local displacement* of the element of string which resides normally at x .

A second inertial observer O' is seen to pass by with speed v . The question arises: How does O' render the physics to which we alluded when we wrote (1); *how does the wave equation transform?* Which we take to mean: How does (1) respond to Galilean transformations

$$\left. \begin{aligned} t &= t' \\ x &= x' + vt' \end{aligned} \right\} \quad (3)$$

The answer falls into our lap when we realize that to describe the “displacement field” O' writes

$$\begin{aligned} \phi(x, t) &= f(x + vt' - ut) + g(x + vt' + ut) \\ &= f(x - [u - v]t') + g(x + [u + v]t') \end{aligned} \quad (4)$$

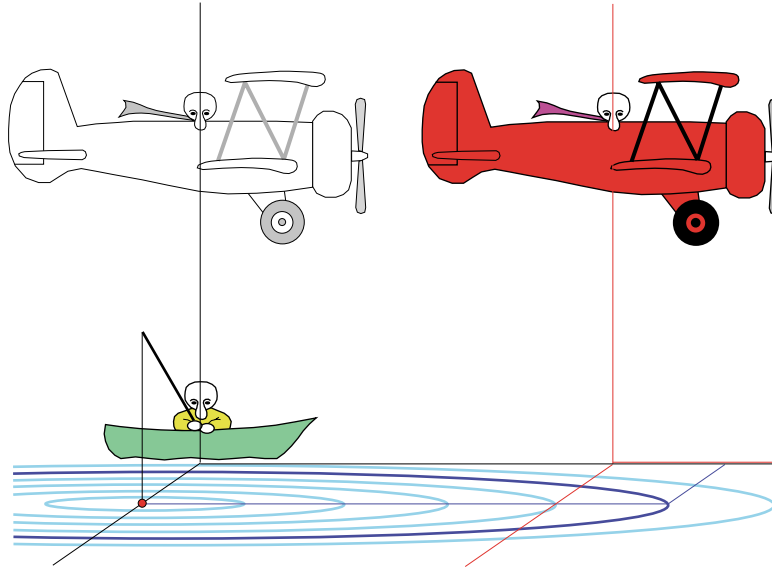


FIGURE 1: 3-dimensional representation of the 1-dimensional issue addressed in the text. We—seated in the boat—are at rest with respect to the water, and see the flying observer to pass by with speed v . In the figure I have attempted to suggest—in anticipation of things to come—that the biplane is very slow: $v < u$, where u denotes the speed (with respect both to us and to the stagnant sea) of wave propagation.

and that such a function cannot possibly satisfy a wave equation of type (1); evidently (4) is in fact a solution of

$$\left[\frac{\partial}{\partial x} + \frac{1}{u-v} \frac{\partial}{\partial t} \right] \left[\frac{\partial}{\partial x} - \frac{1}{u+v} \frac{\partial}{\partial t} \right] \phi(x, t) = 0 \quad (5)$$

which gives back (1) only in the trivial case $v = 0$. Expansion of the differential operator which stands of the left side of (5) gives

$$\underbrace{\left(\frac{\partial}{\partial x} \right)^2 + \frac{2v}{u^2 - v^2} \frac{\partial}{\partial x} \frac{\partial}{\partial t} - \frac{1}{u^2 - v^2} \left(\frac{\partial}{\partial t} \right)^2}_{}$$

which reduces to

$$= \left(\frac{\partial}{\partial x} \right)^2 - \frac{1}{u^2} \left(\frac{\partial}{\partial t} \right)^2 \quad \text{at } v = 0$$

and—interestingly—becomes singular as $v^2 \uparrow u^2$.

The circumstance that (1) and (5) are—though they refer to the same physics—structurally distinct bothers neither O nor O , for both realize that

they stand in asymmetric relationships to the medium: O is at rest with respect to the string; O in motion with respect to the string.

Enter: Young Einstein. I quote now from an English translation of Einstein's "Autobiographical Notes:"¹

"...I came to the conviction that only the discovery of a universal formal principle could lead us to assured results. ... After ten years of reflection such a principle resulted from a paradox upon which I had already hit at the age of 16: if I pursue a beam of light with velocity c ... I should observe such a beam as a spatially oscillatory electromagnetic field at rest. However, there seems to be no such thing, whether on the basis of experience or according to Maxwell's equations. ... It seemed to me intuitively clear that, judged from the standpoint of such an observer, everything would have to happen according to the same laws as for an observer... at rest."

Suppose we had reason to assert the existence in Nature of a class of waves which move *without benefit of a "supporting medium."*² We lose then any grounds on which to tolerate any asymmetry in the relationship of O to O , at least so far as concerns the physics of such waves.

If O sees such a wave³ $\phi(x, t) = f(x - ct) + g(x + ct)$ to satisfy

$$\left[\left(\frac{\partial}{\partial x} \right)^2 - \frac{1}{c^2} \left(\frac{\partial}{\partial t} \right)^2 \right] \phi(x, t) = 0$$

then O must see that wave to satisfy

$$\left[\left(\frac{\partial}{\partial x} \right)^2 - \frac{1}{c^2} \left(\frac{\partial}{\partial t} \right)^2 \right] \phi(x, t) = 0$$

How can such "form invariance" be achieved?

¹ P. A. Schilpp (editor), *ALBERT EINSTEIN: Philosopher-Scientist* (1951), p. 53.

² It is clearly senseless to speak (except poetically) of an "elastic wave in the absence of semi-rigid matter," and until about 1895 it was supposed that electromagnetic waves must, of that same necessity, be supported by an elastic "æther." But experiments designed to detect "motion relative to the æther" [Fizeau (1853), Michelson-Morley (1881-1887), Trouton-Noble (1903)] all yielded null results. It was theoretical desperation which led H. A. Lorentz (~1903) to advance—tentatively—the radical view that electromagnetic waves might get along very well *without* the support of a medium! Like a smile without a face.

³ The notational adjustment $u \rightarrow c$ is intended to emphasize that we have now in mind waves which are "special" (in the sense just described), though not necessarily electromagnetic; put therefore out of mind the thought that c refers to the "velocity of light."

Lorentz transformations. We are forced to the conclusion that to achieve

$$f(x - ct) + g(x + ct) \longrightarrow f(x - ct) + g(x + ct) \quad : \quad \text{all } f(\bullet) \text{ and } g(\bullet)$$

we must presume the relation of O to O to be described *not* by

$$\left. \begin{aligned} t &= t \\ x &= x + vt \end{aligned} \right\} \quad (3)$$

but by some *modified transformation equations*

$$\left. \begin{aligned} t &= \mathcal{T}(x, t; v) \\ x &= \mathcal{X}(x, t; v) \end{aligned} \right\} \quad (6)$$

and, moreover, that these must have the property that they entail

$$x \pm ct = W_{\pm}(x \pm ct) \quad (7)$$

For only then will we have

$$\begin{aligned} f(x - ct) + g(x + ct) &= f(W_-(x \pm ct)) + g(W_+(x \pm ct)) \\ &\equiv f(x - ct) + g(x + ct) \quad : \quad \text{all } f(\bullet) \text{ and } g(\bullet) \end{aligned}$$

Let us now *assume* (6) to possess the *linear* structure characteristic of the Galilean transformation (3), writing

$$\left. \begin{aligned} t &= Pt + px \\ x &= qt + Qx \end{aligned} \right\} \quad (8)$$

where P , p , Q and q depend (in some presently unknown way) upon the kinematic parameter v . From (8) it follows that

$$x \pm ct = (Q \pm cp)x + (q \pm cP)t \quad (9.1)$$

while to achieve compliance with (7) there must exist multipliers $K_{\pm}(v)$ such that

$$= K_{\pm}(v) \cdot (x \pm ct) \quad (9.2)$$

From (9) we are led to a quartet of equations

$$\begin{aligned} Q + cp &= K_+ \\ Q - cp &= K_- \\ q + cP &= +cK_+ \\ q - cP &= -cK_- \end{aligned}$$

from which it follows readily that

$$\left. \begin{aligned} P &= \frac{c}{2c}(K_+ + K_-) \\ p &= \frac{1}{2c}(K_+ - K_-) \\ Q &= \frac{1}{2}(K_+ + K_-) \\ q &= \frac{c}{2}(K_+ - K_-) \end{aligned} \right\} \quad (10)$$

But the functions $K_{\pm}(v)$ and the relation of $c(v)$ remain at present unknown.

To make further progress, let us REQUIRE that the transformation which sends

$$O \xrightarrow[v]{} O$$

be *symmetric* in the sense that $v \rightarrow -v$ achieves its inversion; then

$$\begin{aligned} x \pm ct &= K_{\pm}(v) \cdot (x \pm ct) \\ x \pm ct &= K_{\pm}(-v) \cdot (x \pm ct) \end{aligned}$$

supplies the information that

$$K_+(v) \cdot K_+(-v) = K_-(v) \cdot K_-(-v) = 1 \quad (11.1)$$

Note also that time-reversal sends

$$\begin{array}{c} x \pm ct = K_{\pm}(+v) \cdot (x \pm ct) \\ \downarrow \text{time-reversal} \\ x \mp ct = K_{\pm}(-v) \cdot (x \mp ct) \end{array}$$

so if we REQUIRE *time-reversal invariance* we are led to the condition

$$K_{\pm}(+v) = K_{\mp}(-v) \quad (11.2)$$

Equations (11) conjointly entail

$$K(v) \equiv K_+(v) = \frac{1}{K_-(v)} \quad (12)$$

which serves to reduce the number of unknown functions.

Consider finally how O describes O 's “clock-at-himself,” and *vice versa*. Working from (8) and (10), we find that

$$(t, vt) \longleftarrow (t, 0) \quad \text{entails} \quad q/P = v = c \frac{K_+ - K_-}{K_+ + K_-} \quad (13.1)$$

$$(t, 0) \longrightarrow (t, -vt) \quad \text{entails} \quad q/Q = v = c \frac{K_+ - K_-}{K_+ + K_-} \quad (13.2)$$

which upon comparison give

$$c(v) = c \quad : \quad \text{all } v \quad (14)$$

The striking implication is that

If O and O “share the wave equation” then—granted certain natural assumptions—they must necessarily “share the value of c .”

It is on this basis that henceforth we abandon the red c black c distinction. Introducing the dimensionless relative velocity parameter

$$\beta \equiv v/c$$

and returning with (12) to (13), we obtain

$$\beta = \frac{K^2 - 1}{K^2 + 1} \quad \text{whence} \quad K(v) = \sqrt{\frac{1 + \beta}{1 - \beta}} \quad (15)$$

All the formerly “unknown functions” have now been determined.

Bringing this information to (10), and returning with the results to (8), we obtain

$$\left. \begin{aligned} t &= \frac{1}{2} \left[\sqrt{\frac{1+\beta}{1-\beta}} + \sqrt{\frac{1-\beta}{1+\beta}} \right] t + \frac{1}{2c} \left[\sqrt{\frac{1+\beta}{1-\beta}} - \sqrt{\frac{1-\beta}{1+\beta}} \right] x = \gamma (t + vx/c^2) \\ x &= \frac{c}{2} \left[\sqrt{\frac{1+\beta}{1-\beta}} - \sqrt{\frac{1-\beta}{1+\beta}} \right] t + \frac{1}{2} \left[\sqrt{\frac{1+\beta}{1-\beta}} + \sqrt{\frac{1-\beta}{1+\beta}} \right] x = \gamma (vt + x) \end{aligned} \right\} \quad (16)$$

where

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \dots$$

Transformations $(t, x) \rightarrow (t, x)$ of the design (16) first presented themselves when Lorentz looked (~ 1904) looked to the transformation-theoretic properties (not of the wave equation but) of Maxwell’s equations, and are called *Lorentz transformations*. For $v \ll c$ (equivalently: in the formal limit $c \uparrow \infty$) (16) gives

$$\begin{aligned} t &= t + \dots \\ x &= x + vt + \dots \end{aligned}$$

which exposes the sense in which the Lorentz transformations

- enlarge upon
- reduce to
- contain as approximations

the Galilean transformations (3).

We have achieved (16) by an argument which involves little more than high school algebra—an argument which approaches the masterful simplicity of Einstein’s own line of argument. But Einstein’s argument—which with its population of idealized trains, lanterns and meter sticks seems to me to read more like “mathematical epistemology” than physics—remains unique in the field; I do not claim to understand it well enough to be able to reproduce it in the classroom, do not find it particularly compelling, and do not know how seriously it today is to be taken (since relativity has been found to pertain at a scale so microscopically fine as to render Einstein’s “thought experiments” meaningless). My own argument, on the other hand, springs from a question—*How does this object of interest transform, and what transformations preserve its form?*—which in its innumerable variants has been fruitfully standard to mathematical physics for at least 250 years.

Theoretical status of the Lorentz transformations. A world in which inertial observers O and O' use

$$\left. \begin{aligned} t &= t' \\ x &= x' + vt' \end{aligned} \right\} \quad (3)$$

when comparing mechanical observations, but must use

$$\left. \begin{aligned} t &= \gamma(t' + vx'/c^2) \\ x &= \gamma(vt' + x') \end{aligned} \right\} \quad (16)$$

when comparing observations pertaining to “mediumless waves” (and must, moreover, be prepared to assign distinct values c', c'', \dots to c when confronted with distinct systems of such waves) is a world which is *theoretically unhinged*—a world in which the physics books can, in their totality, pertain to the experience of (at most) a single observer. Such a state of affairs would be inconsistent with the spirit of the Copernican revolution.

It follows—on grounds which, whether formal/philosophical/æsthetic, are clearly fundamental—that not more than one of the options spelled out above can figure in a comprehensive physics. How to proceed?

- We might try to stick with (3); we then retain Newtonian dynamics intact, but must give up the notion of a “mediumless wave.” This may seem acceptable on its face, but entails that we also abandon Maxwellian electrodynamics and the associated electromagnetic theory of light—theories which conform very well to observation.
- We might, alternatively, adopt some instance of (16)—namely, the instance which results from promoting some c -value to the status of a *universal constant of Nature*. If, in particular, we set

$$c = \text{velocity of light}$$

then we retain Maxwellian electrodynamics intact, but must abandon Newtonian dynamics, which becomes merely the leading low-velocity *approximation* to a “relativistic dynamics.” This, clearly, is the more “interesting” way to go. We are led thus—with Einstein (1905)—to postulate the

PRINCIPLE OF RELATIVITY: Physical formulæ and concepts shall be “admissible” if and only if they are form-invariant with respect to the Lorentz transformations (16).

Note that the Principle of Relativity refers to no specific physical phenomenon; it refers, instead, to the necessary *structure of physical theories in general*. It stands to physics in much the same relationship that the Rules of Syntax (which refer to no specific utterance, but to the design of “well-formed sentences”) stand to language.

Geometrical consequences of the Lorentz transformation formula. Diagrams such as Figure 2 are commonly encountered already in pre-relativistic physics, where they are used to represent kinematic/dynamic events. The point to which

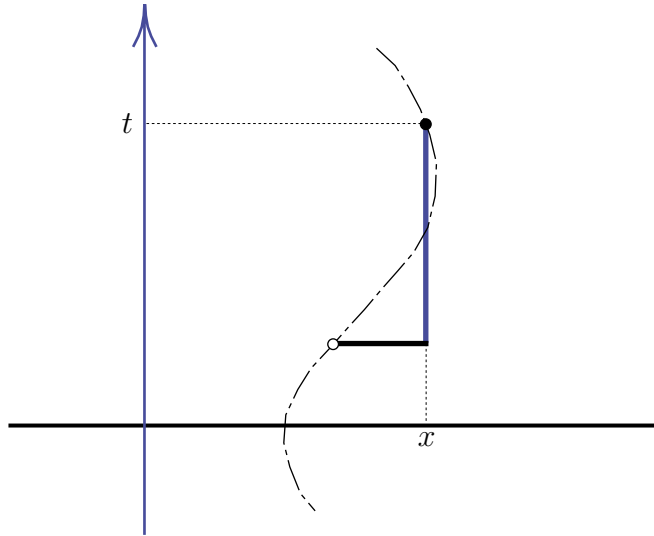


FIGURE 2: *Space-time representation of the motion $x(t)$ of a point (kinematics) or particle (dynamics). The diagram is in effect a movie—a t -parameterized stack of “time-slices.” In pre-relativistic physics it makes one kind of good sense to speak of the Euclidean distance between two space points (as measured by a meter stick), and quite a different kind of good sense to speak of the temporal interval separating two time-slices (as measured by a clock), but no sense to speak of “the distance between two events,” \circ and \bullet .*

I would draw attention is that such “space-time diagrams” arise by formal fusion of two distinct notions:

- a 1-dimensional “time axis,” the points of which are well-ordered, and on which intervals have the physical dimension of TIME;
- an N -dimensional “space,” the points of which are (except in the case $N = 1$) *not* well-ordered, and on which intervals have the physical dimension of LENGTH.

Notice now that, upon returning with (12) to (9.2), we have

$$(x \pm ct) = K^{\pm 1}(x \pm ct) \quad (17)$$

from which it follows that

$$(ct)^2 - x^2 = (ct)^2 - x^2 \quad (18)$$

This last equation establishes the sense in which

the expression $(ct)^2 - x^2$ is Lorentz-invariant

just as, on the Euclidean plane,

the expression $x^2 + y^2$ is rotationally invariant

In the latter context, $r^2 \equiv x^2 + y^2$ defines the (squared) “Pythagorean length” of the spatial interval separating the point (x, y) from the origin $(0, 0)$; it is from that definition that the Euclidean plane—now a “metric space”—acquires its distinctive metric properties.

Which brings us to the work of Hermann Minkowski, who had been one of Einstein’s teachers (of mathematics) at the ETH in Zurich, and who in 1909 wrote

“The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.”

Minkowski’s idea was to let $s^2 \equiv (ct)^2 - x^2$ serve to *define* the (squared) “Minkowskian length” of the spacetime interval separating the “event” (ct, x) from the origin $(0, 0)$. But this simple means (see Figure 3) did he achieve

$$\text{space} \otimes \text{time} \longrightarrow \text{spacetime}$$

and thus expose the deeper significance of Einstein’s accomplishment (which Einstein himself had somehow failed to notice).

At this point it becomes natural to make a notational adjustment, writing

$$\left. \begin{array}{l} x^0 \text{ in place of } ct \\ x^1 \text{ in place of } x \end{array} \right\} \text{ both have the dimensionality of LENGTH}$$

and

$$\begin{aligned} s^2 &= (x^0)^2 - (x^1)^2 && (19) \\ &\downarrow \\ &= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \quad \text{in 4-dimensional spacetime} \end{aligned}$$

It is, of course, the availability of the dimensioned constant c which makes such an adjustment possible; i.e., which makes it possible to “measure time in centimeters.” The right side of (19) is precisely Pythagorean *except for the funny sign*, which makes a world of difference.

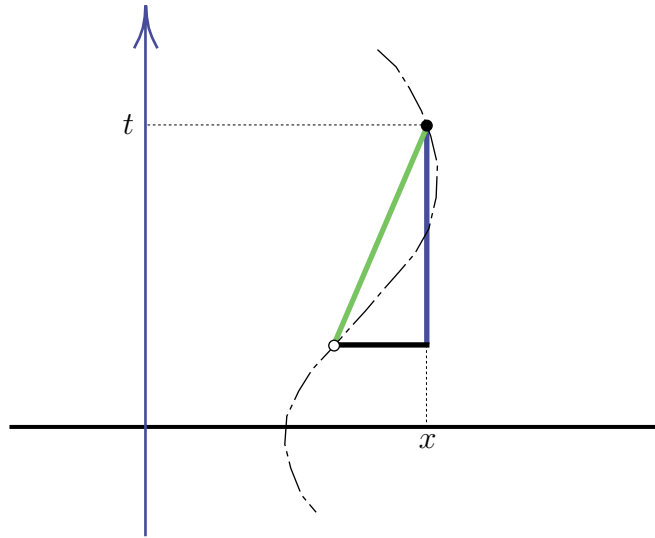


FIGURE 3: By assigning meaning to the (squared) length of the green interval

$$s^2 \equiv c^2(t - t_0)^2 - (x - x_0)^2$$

Minkowski invented “spacetime”—the $(3 + 1)$ -dimensional metric continuum which serves as the blackboard upon which all physics is written, the arena of physical experience. The dashed curve is, in relativistic parlance, called the “worldline” of the point/particle.

It is intuitively evident that if, on the Euclidean plane, we subject the point (x, y) to all possible origin-preserving rotations (rotations about the origin, as we might accomplish with a simple compass) we will be led to the set of points (x, y) which satisfy

$$x^2 + y^2 = x^2 + y^2 \equiv r^2$$

This, in Euclidean geometry, is the equation of the *isometric circle* through (x, y) —the set of all points which lie at the (squared) Pythagorean distance r^2 from the origin.

Similarly: if, in 2-dimensional spacetime, we subject the event (x^0, x^1) to all possible (origin-preserving) Lorentz transformations (16)—i.e., if we look to the coordinates (x^0, x^1) which all possible inertial observers assign to that event—we will be led to the equation

$$(x^0)^2 - (x^1)^2 = (x^0)^2 - (x^1)^2 \equiv s^2$$

which serves to inscribe on spacetime the *isometric hyperbola* through (x^0, x^1) , the set of all events which lie at the same (squared) *Minkowskian* distance s^2 from the origin. Notice, however, that while $r^2 \geq 0$ in all cases, the sign of s^2

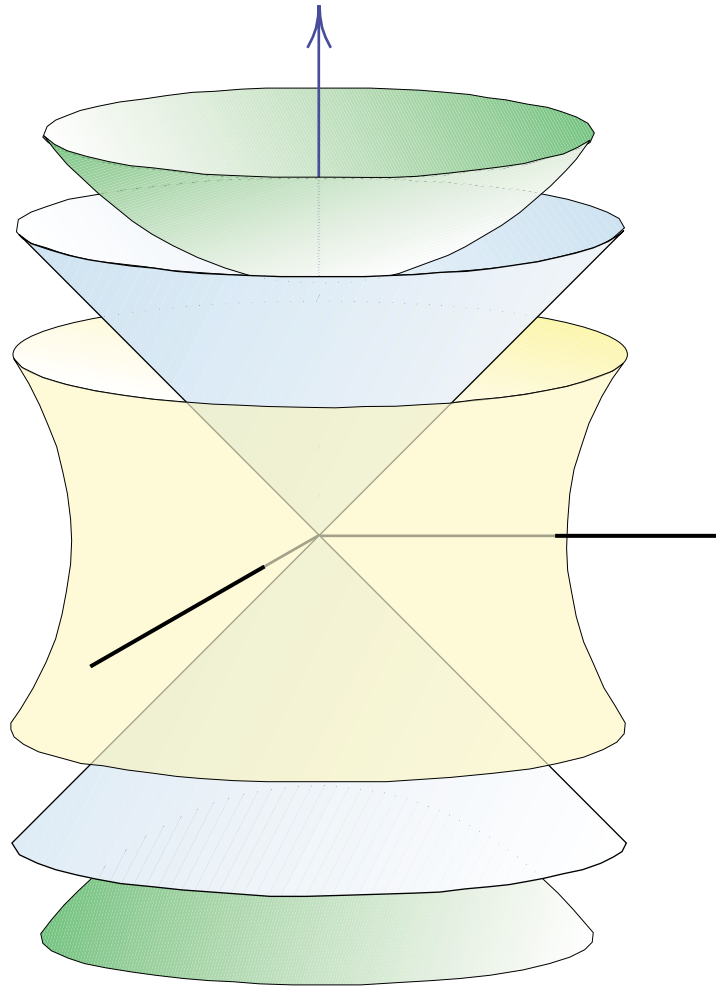


FIGURE 4: In $(2 + 1)$ -dimensional spacetime the lightcone (set of points at zero distance from the origin)—blue in the figure—is concentric about the time-axis. Points whose separation from the origin is time-like lie interior to the cone, points whose separation is spacelike lie exterior. Isometric surfaces are “hyperboloids of two sheets” (up/down-turned green bowls in the figure) in the time-like case, “hyperboloids of a single sheet” (yellow girdle) in the space-like case.

is indefinite; one says that (x^0, x^1) is separated from the origin by

- a *time-like* interval if $s^2 > 0$: $(x^0, 0)$ provides a set of examples;
- a *light-like* (or null) interval if $s^2 = 0$: necessarily $x^1 = \pm x^0$ in such cases;
- a *space-like* interval if $s^2 < 0$: $(0, x^1)$ provides a set of examples.

The preceding figure provides a geometrical interpretation of those distinctions,

and makes clear the sense in which the (Minkowskian) geometry of spacetime is *hyperbolic*.

Lorentz transformations cause the points on each isometric surface to circulate amongst themselves, but each such surface—viewed as a whole—is *invariant* under such transformation. Which is to say: Figure 4 is a shared possession of *all* inertial observers. We are brought thus to this alternative formulation of the Principle of Relativity:

The well-formed formulæ of physics transform in such a way as to mimic the transformation properties of the spacetime upon which that physics is inscribed. All other formulæ refer to “impossible physics,” and must be discarded (except insofar as they may retain utility as approximations to relativistic statements).

How did the “speed of light” acquire such importance? The short answer, in my view, is this: It didn’t. The constant c refers more fundamentally to *the limiting speed with which interactive news/influences—of whatever nature—can propagate*. Though absent from the approximate equations which are the stuff of “non-relativistic physics,” it enters either explicitly or covertly into *all* the equations of physics, and in relatively rare instances appears nakedly, as “the speed of something”—typically a wave, or (in quantum mechanics) a “massless particle” (photon, ...).

I used to hold that to call c the “speed of light” was unfair to neutrinos; now that neutrinos have been revealed to have a small mass (and therefore to travel with speeds always less than c) I hold that such terminology is simply unfair to the deeper facts of the matter. It is for this reason that I, in my more pedantic moments, prefer to speak not of the “speed of light” but of the “natural limiting velocity,” the “speed of the mail.”

When viewed in this light, the question posed above invites reformulation: What fundamental principles of elementary particle physics serve to foist electrodynamics—light—upon us as a fact of Nature? Why, in short, does light exist, and why does it happen to move with speed c ? The answer remains unknown. But this much is clear: even if light were expunged from reality, the constant c would still have plenty of work to do (though sentient creatures—physicists—able to ponder the matter would, by that adjustment, almost certainly have been rendered impossible).

Relativity emerges when (i) one posits the *existence* of a limiting velocity, and (ii) stipulates that its value c is *agreed upon by all inertial observers* (of which, we learned from Galileo/Newton, there exist not just one but a continuum).

That literal lanterns and lightbeams are extraneous to the fundamentals of Einstein’s epistemological argument would have been granted by Einstein himself, had it emerged that the photon is in fact not exactly massless (i.e., that light does not actually travel with speed c). Einstein worked from a particular

instance of a “physically exposed naked c ,” but could in principle have worked from any other such instance. . . though no natural candidate presents itself, and light is common stuff. Similarly. . .

I worked from properties of the wave equation—common stuff, relating to physics directly evident to the senses—but had I worked from *any* particular instance of a Lorentz-covariant equation I would have been led back (ultimately, and probably less simply) to the Lorentz transformations, and to the implied geometrical structure of spacetime.

Algebraic/geometric commentary. I have now completed the discussion to which my title refers, but to serve my anticipated readers must sketch how one gets from where we are to the relativity of the textbooks; it is to that assignment that I now turn.

If spacetime were 2-dimensional then in place of Figure 4 one would draw Figure 5. Most but not all of the most characteristic essentials of relativity survive intact the dimensional reduction $2 \leftarrow 4$; one loses only those aspects of relativity which relate specifically to spatial *rotations*—aspects of the subject which Einstein himself missed, and which textbooks typically omit. I embrace

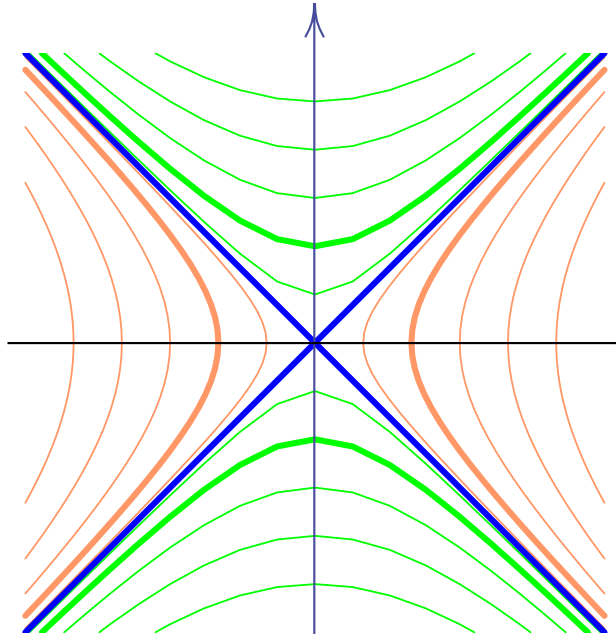


FIGURE 5: *The figure which would replace Figure 4 if spacetime were 2-dimensional. This is, in effect, a cross-section of Figure 4 (set $x^2 = 0$), which is itself a cross-section (set $x^3 = 0$) of a more physical “4-dimensional figure;” such figures are impossible to draw, but easy to display as movies (animations of Figure 4).*

the fiction that “spacetime is 2-dimensional” because of the many diagrammatic opportunities made thus available.

The equations

$$\left. \begin{aligned} t &= \frac{1}{2} \left[\sqrt{\frac{1+\beta}{1-\beta}} + \sqrt{\frac{1-\beta}{1+\beta}} \right] t + \frac{1}{2c} \left[\sqrt{\frac{1+\beta}{1-\beta}} - \sqrt{\frac{1-\beta}{1+\beta}} \right] x = \gamma (t + vx/c^2) \\ x &= \frac{c}{2} \left[\sqrt{\frac{1+\beta}{1-\beta}} - \sqrt{\frac{1-\beta}{1+\beta}} \right] t + \frac{1}{2} \left[\sqrt{\frac{1+\beta}{1-\beta}} + \sqrt{\frac{1-\beta}{1+\beta}} \right] x = \gamma (vt + x) \end{aligned} \right\} \quad (16)$$

and

$$\left. \begin{aligned} x + ct &= \sqrt{\frac{1+\beta}{1-\beta}} (x + ct) \\ x - ct &= \sqrt{\frac{1-\beta}{1+\beta}} (x - ct) \end{aligned} \right\} \quad (17)$$

provide algebraically equivalent descriptions of the Lorentz transformation

$$\Lambda(\beta) : (t, x) \leftarrow (t, x)$$

but the latter—which can be notated

$$\left. \begin{aligned} (x^0 + x^1) &= K^{+1}(\beta) \cdot (x^0 + x^1) \\ (x^0 - x^1) &= K^{-1}(\beta) \cdot (x^0 - x^1) \end{aligned} \right\} \quad (20)$$

—is, in an obvious sense, simpler. Evidently the coordinates

$$\begin{aligned} \xi^+ &\equiv x^0 + x^1 \\ \xi^- &\equiv x^0 - x^1 \end{aligned}$$

are, in some respects, better suited to the discussion of relativistic matters than are the physical variables t and x (equivalently: x^0 and x^1). The equations

$$\xi^+ = \text{constant} \quad , \quad \xi^- = \text{constant}$$

would, if their graphs were superimposed upon Figure 5, yield lines which slope down/up at 45°; such coordinates are for this reason sometimes said to be “lightlike.” Some authors prefer to make (20) their analytical point of departure, and have given the name “K-calculus” to the resulting body of theory, but this practice, though apparently commonplace in England,⁴ has never gained many adherents in the United States.

Some sense of the power latent in the K-calculus (which it is not my business to promote) can be gained from consideration of this question: How does one describe the result of *successive* Lorentz transformations? How, in

⁴ See Hermann Bondi, *Relativity and Common Sense*” *A New Approach to Einstein* (1962) pp. 116–119, or Chapter 26 in David Bohm, *The Special Theory of Relativity* (1965).

short, do Lorentz transformations *compose*? The answer can be extracted with labor from (16), but is made immediately evident by (17): solving

$$\sqrt{\frac{1+\beta_2}{1-\beta_2}} \sqrt{\frac{1+\beta_1}{1-\beta_1}} = \sqrt{\frac{1+\beta}{1-\beta}}$$

for β , we obtain

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \quad (21)$$

which is familiar as the relativistic “velocity addition formula,”⁵ and entails

$$\Lambda(\beta_2) \cdot \Lambda(\beta_1) = \Lambda\left(\frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}\right) \quad (22)$$

Evidently

- $\Lambda(0)$ describes the do-nothing transformation;
- $\Lambda(-\beta)$ describes the inverse of $\Lambda(\beta)$.

We conclude that the set $\{\Lambda(\beta)\}$ of Lorentz transformations possesses the “group property,” and that (22) is simply the “composition law” characteristic of that group. The inexhaustibly rich ramifications of this elementary remark (actually its 4-dimensional counterpart) permeate all of relativistic physics, and become especially conspicuous in the theory of elementary particles.

It is sometimes useful to notice that

$$K(\beta) \equiv \sqrt{\frac{1+\beta}{1-\beta}} = \gamma(1 + \beta)$$

Cognoscenti will recognize the factors K^\pm to be precisely the eigenvalues of the “Lorentz matrix”

$$\mathbb{L}(\beta) \equiv \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$

in terms of which (17) can be notated

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \mathbb{L}(\beta) \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

I display now a gallery of figures intended to illustrate some of the geometrical implications of (20); i.e., to assign geometrical interpretations to some aspects of the theory of Lorentz transformations.

⁵ It becomes fairly natural at this point to introduce the so-called “rapidity” parameter $\psi \equiv \tanh \beta$, in terms of which (21) reads

$$\begin{aligned} \tanh \psi &= \frac{\tanh \psi_1 + \tanh \psi_2}{1 + \tanh \psi_1 \tanh \psi_2} \\ &= \tanh(\psi_1 + \psi_2) \quad \text{by a familiar identity} \end{aligned}$$

In relativity the simple “velocity addition law” is lost—replaced by an equally simple “rapidity addition law.”

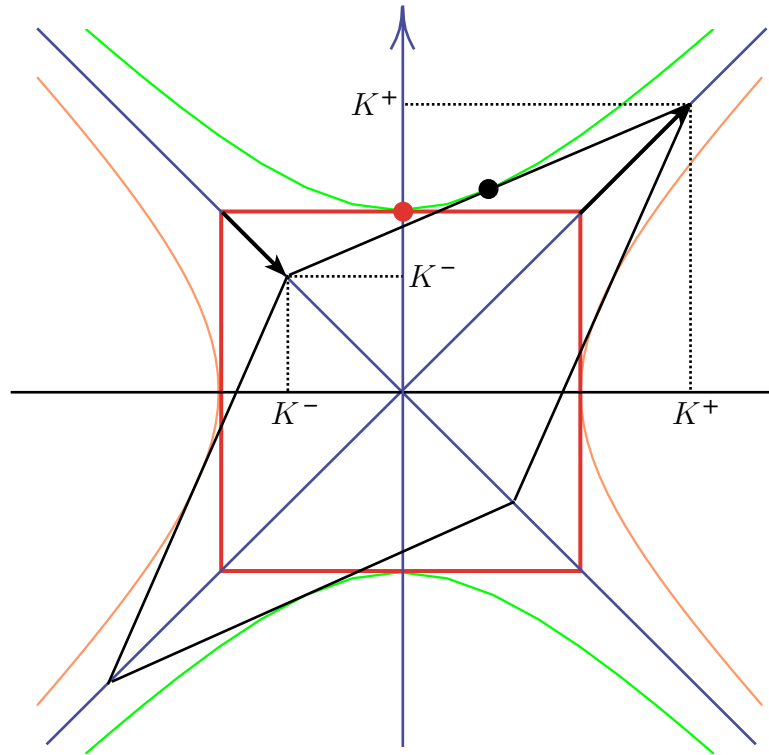


FIGURE 6: \circ is thinking of a set of events which serve collectively to inscribe a unit square on spacetime. Equation (20) describes the Lorentz transforms of the vertices of the square, and from the linearity of the transformation rule (it sends lines to lines) we can infer what happens to the sides: the square becomes a parallelogram (of—curiously—identical Euclidean area). The red dot marks the point of tangency of the top side with an isometric curve. Its image (black dot) serves similarly to mark the point of tangency of the transformed top side with that same isometric curve; this observation illustrates the force of a previous remark: “Lorentz transformations cause the points on each isometric curve/surface to circulate amongst themselves, but each such curve/surface—viewed as a whole—is invariant under such transformation.” The figure makes plain also the sense in which the factors K^\pm acquire meaning as the eigenvalues of the transformation, the factors by which \nearrow and \nwarrow vectors are rescaled.

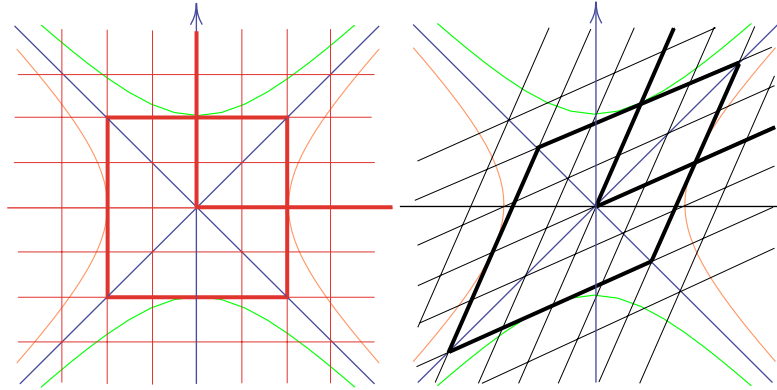


FIGURE 7: *Elaboration of the preceding figure. \mathcal{O} has inscribed a Cartesian gridwork on spacetime. On the right is shown the Lorentz transform of that coordinate grid. John Wheeler has referred in this connection to the “collapse of the egg crate,” though that picturesque terminology is somewhat misleading: egg crates preserve side-length when they collapse, while the present mode of collapse preserves Euclidean area. Note that tangents to isometric curves remain in each case tangent to the same such curve. The entire population of isometric curves (Figure 5) can be recovered as the population of envelopes of the grid lines, as generated by allowing β to range over all allowed values ($-1 < \beta < +1$). Conversely, we can reconstruct Figure 7 by drawing parallel tangents to isometric curves. Figures 5 & 7 are in that sense equivalent, and—though I do not advocate such a procedure—one could consider the Minkowskian geometry of spacetime to be implicit in a “principle of area-preserving egg crate collapse.”*

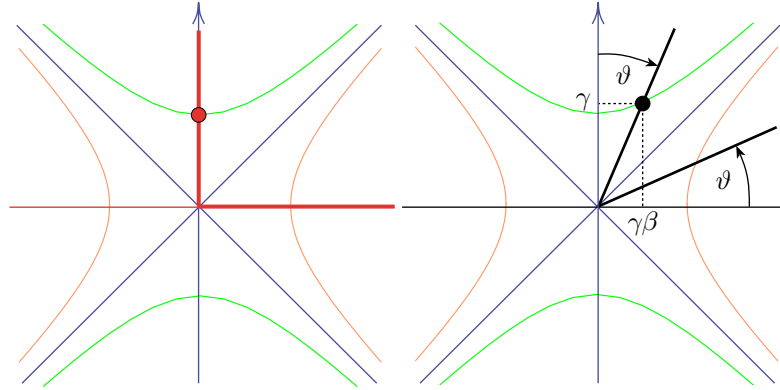


FIGURE 8: O writes $(ct, 0)$ to describe the “ t^{th} tick of his clock.” Working from (16) we find that O assigns coordinates $(\gamma t, \gamma\beta t)$ to those successive events. The implication is that the (Euclidean) angle ϑ subtended by

- O 's time axis and
- O 's representation of O 's time axis

can be described

$$\tan \vartheta = \beta$$

The same angle, by a similar argument, arises when one looks to O 's representation of O 's space axis. In the preceding caption I mention that, taking Figure 5 as one's point of departure, one could recreate the instance of Figure 7 which is appropriate to any prescribed β -value. But to execute such a program one needs the information just obtained.

Interpretive commentary. In O 's estimation the events $(0, 0)$ and $(ct, 0)$ have temporal separation t . But—as was argued in connection with the preceding figure— O assigns coordinates $(0, 0)$ and $(ct, x) = (\gamma t, \gamma\beta t)$ to those events, and reports their *temporal* separation to be

$$t = \gamma t \geq t \tag{23}$$

This is “time dilation” in a nutshell: time-intervals are seen by observers in relative motion to be prolonged; the clocks of observers in flight are seen to run slow.

The next pair of figures refer to the origin/meaning of a complementary effect—“Lorentz contraction”—which are relatively more subtle:

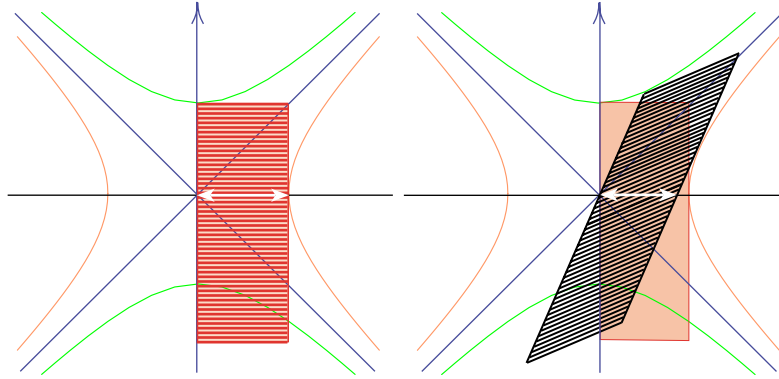


FIGURE 9: The figure on the left shows O 's representation of his meter stick just sitting there, sitting there, sitting there... The figure on the right provides O 's representation of that same set of events, and the arrow indicates the meaning (spatial separation of the ends at any given instant) which O ascribes to the perceived length of O 's stick. It is diagrammatically evident that (and why)

$$\text{length } \ell \text{ ascribed by } O \leq \text{length } \ell \text{ ascribed by } O'$$

but to obtain a quantitatively precise statement we must engage in a little analytical geometry, as undertaken in connection with the next figure.

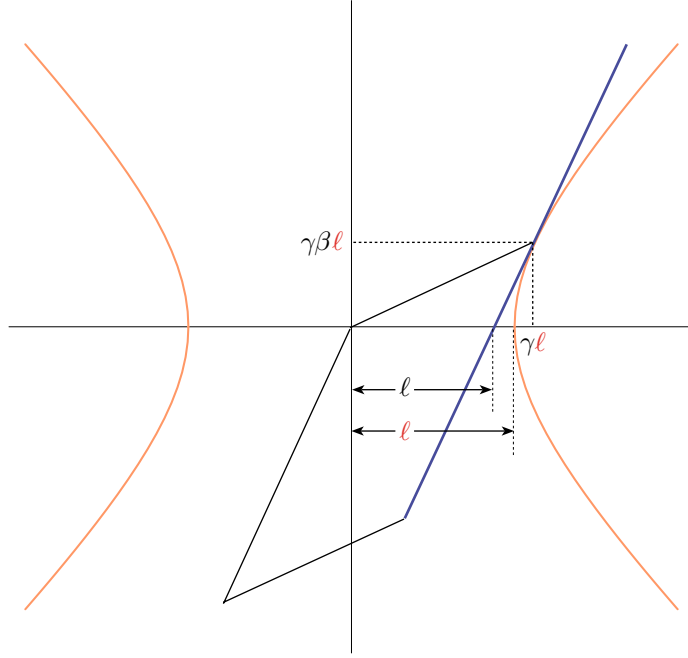


FIGURE 10: Let us, for the purposes of this discussion, agree to write y for x^0 , x for x^1 . The spacelike isometric hyperbola can in this notation be described

$$x^2 - y^2 = \ell^2$$

By implicit differentiation we have $2x - 2y \frac{dy}{dx} = 0$, giving

$$\frac{dy}{dx} = x/y$$

Lorentz transformation sends $(0, \ell) \rightarrow (\gamma\beta\ell, \gamma\ell)$, both of which lie on the isometric curve. The blue tangent at the latter point has

$$\text{slope} = \gamma\ell / \gamma\beta\ell = 1/\beta$$

From the figure we read

$$\gamma\beta\ell = (\ell - \ell) \cdot \text{slope}$$

which after a little algebra gives $\ell = (1 - \beta^2)\gamma\ell$ or

$$\ell = \gamma^{-1}\ell \tag{24}$$

We conclude that the length ascribed to a moving meter stick is reduced by the precisely the reciprocal of the time-dilation factor.

Here a cautionary word is in order: one should be careful not to confuse the result just obtained with an account of how rapidly moving objects would *appear* if *photographed* (on fast color film). This was first pointed out by R. Penrose⁶ and T. Terrell⁷ in the late 1950's, and led V. Weisskopf, in a frequently cited popular article,⁸ to express his amazement that the world's physicists could have overlook such an elementary point for more than fifty years!⁹

Relativity imposes an unfamiliar restriction upon the conditions under which it becomes meaningful to say that “event A preceded event B,” and therefore upon the circumstances under which it becomes possible to speak (however informally) of A “causing” B. This far-reaching development arises from the simple geometrical circumstance illustrated in Figure 11.

Perhaps even more profoundly consequential is the *breakdown of the concept of distant simultaneity*, which arises from an equally elemental circumstance (Figure 12). So radical was this development that it rendered instantly untenable (except as a useful approximation) one of the crown jewels of Newtonian physics—the Universal Law of Gravitation. For the latter posits “instantaneous action at a distance,” a concept to which many of Newton's continental contemporaries took exception already on philosophical grounds, and which, though meaningful to each individual inertial observer, can have no *shared* meaning, and is therefore inconsistent with the Principle of Relativity.

Interaction, if prevented from being remote, must perforce be local, having the character of a *collision*. Statements of the form “A collided with B at the spacetime point (x^0, x^1) ” are simultaneously meaningful to *all* inertial observers. The remote interactions observed in Nature must therefore be *intermediated*, by an infinite sequence of click/click/click events. The intrusion of c into physics can be understood as having to do with the limiting speed with which such click/click/click elbow-jabbing can be passed along (which can in turn be understood as having to do with “how fast God can solve the associated equations”?).

When we look to the physical locus of all that (invisible) intermediating click/click/click activity we find ourselves virtually forced to introduce the “field” concept into our physics...in which connection we recall that it was from electrodynamics—the theory of charged particle interactions mediated by the electromagnetic field—that Einstein was, in the first instance, *led* to relativity.¹⁰

⁶ “The apparent shape of a relativistically moving sphere,” Proc. Camb. Phil. Soc. **55**, 137 (1959).

⁷ “Invisibility of Lorentz contraction,” Phys. Rev. **116**, 1041 (1959).

⁸ Physics Today (September 1960).

⁹ For an accessible account of the curious details, see G. Scott & H. van Driel, “Geometric appearances at relativistic speeds,” AJP **38**, 971 (1970).

¹⁰ Einstein's original paper bore the title “On the electrodynamics of moving bodies” (“Zur Elektrodynamik bewegter Körper,” Annalen der Physik **17**, (1905)). It ran (in translation) to twenty-nine pages, and had no figures.

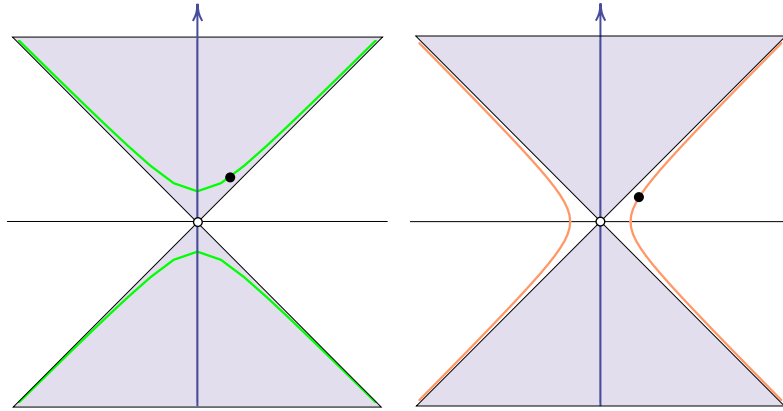


FIGURE 11: *On the left, an event \bullet is seen by O to lie within the lightcone with vertex at \circ (which is to say: the interval separating bullet from \circ is timelike), and to occur after \circ . Other observers place \bullet at other points on the same isometric curve, but all agree that \bullet lies in the future of \circ . On the right, O considers \bullet (exterior to the lightcone) to have occurred after \circ , but other observers will see a reversed temporal order. The moral: temporal sequence has an observer-independent meaning only interior to the lightcone; it becomes possible in principle to say that “ A caused B ” only if A lies within the lightcone which extends forward from B .*

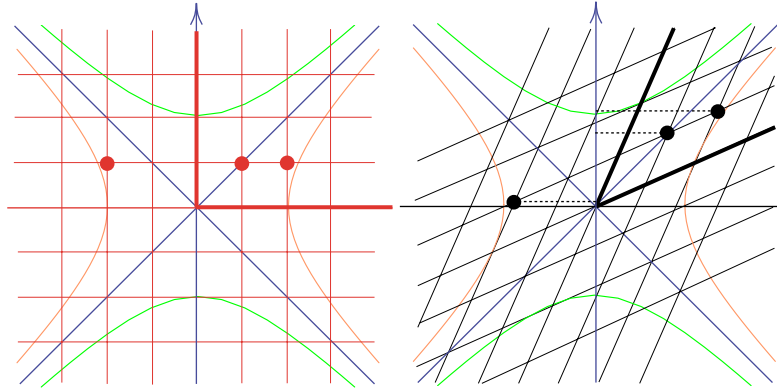


FIGURE 12: *Inertial observer O sees three spatially separated events to occur simultaneously. O , on the other hand, sees them to occur at distinct times. Since distinct observers can assign no shared meaning to “remote simultaneity,” relativistic physics must proceed without reference to such a concept. It must therefore proceed without reference also to Newton’s notion of “instantaneous action at a distance.”*

Optical Doppler effect. Theory of the familiar pre-relativistic Doppler effect (as it relates, for example, to acoustics) is a play with three actors:

- the source (considered to be at rest with respect to observer O),
- the medium, and
- the detector (considered to be at rest with respect to observer O).

The theory of light, however, proceeds *without* the support of a medium, is therefore necessarily relativistic, and leads to Doppler formulæ distinct from those encountered in acoustics. Working from the upper shaded sector on the left side of Figure 13, we consider the following scenario:

O and O are initially coincident. O sets off with speed v . At time τ the stay-at-home observer O emits an optical signal. At what time t does O see the signal to reach O ? From

$$vt = c(t - \tau)$$

O computes

$$t = \frac{1}{1-\beta} \tau$$

O , on the other hand, sees himself to be hit by the signal at time

$$\tau = \frac{1}{\gamma} t = \sqrt{\frac{1+\beta}{1-\beta}} \tau$$

Had O seen O to be *approaching* with speed v then the same line of argument (lower shaded sector) would have given $t = \frac{1}{1+\beta} \tau$ whence

$$\tau = \frac{1}{\gamma} t = \sqrt{\frac{1-\beta}{1+\beta}} \tau$$

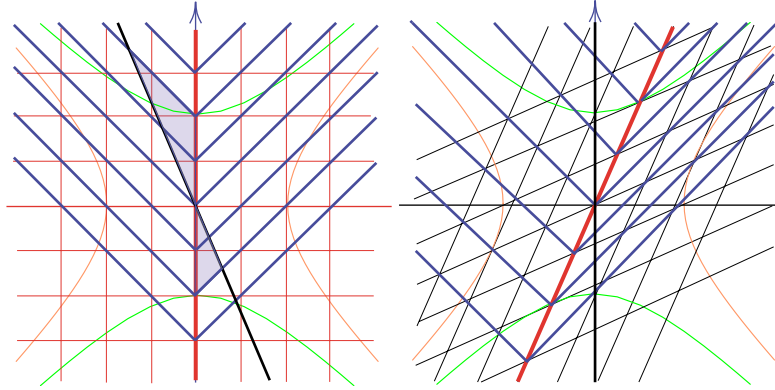


FIGURE 13: On the left, O —who is broadcasting optical signals with period τ —sees O to pass by with speed $v < c$. On the right is O 's representation of that same set of events. It is diagrammatically evident that O 's pulse-interception period is

$$\tau_{\text{approaching}} < \tau < \tau_{\text{receding}}$$

The results thus obtained are usually expressed not as a period shift but as a frequency shift:

$$\nu_{\text{approaching}} = \sqrt{\frac{1+\beta}{1-\beta}} \nu > \nu > \nu_{\text{receding}} = \sqrt{\frac{1-\beta}{1+\beta}} \nu \quad (25)$$

In astrophysical/cosmological applications (where ν_{receding} is the frequency *observed*) it proves often more convenient to speak of the “red-shift parameter”

$$z \equiv \frac{\nu_{\text{source}} - \nu_{\text{observed}}}{\nu_{\text{observed}}} = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \sim \beta + \frac{1}{2}\beta^2 + \dots \quad (26)$$

The formulæ (23) and (24) which serve respectively to describe time dilation and Lorentz contraction incline one to suppose that “if, in relativity, an expression scales up/down when transformed, then the scale-up factor is γ and the scale-down factor is $1/\gamma$.” We learn from (25) and (26) that that frequently handy rule of thumb is in fact not invariably correct: in the latter equations we have encountered not γ -factors but what amount (see again (15)) to “naked K -factors.”

Retirement of the “clock on God’s wall” and of some inertial observers. Newton found it natural to assume that all observers—whatever their state of motion—experience time identically; i.e., that all are able and content to read the time

from the same “clock on God’s wall.” It was on that postulated basis that at (3) we wrote $t = \bar{t}$.

In relativity one imagines all observers to be equipped with identical Rolexes, and to carry identical standard-issue meter sticks, and to be each quite vividly aware of the conceptual distinction between “time” and “space,” but is nevertheless forced to the conclusion that the coordinates (t, x) , (\bar{t}, \bar{x}) , . . . which observers O , \bar{O} , . . . assign to an event lack any such universality feature, that the transformation

$$(t, x) \leftarrow (\bar{t}, \bar{x})$$

involves an *intermixing* of space and time coordinates, as described by (16) and illustrated in most of the recent figures. The clock on God’s wall has been consigned to the intellectual junk heap.

This development is relatively innocuous as it relates to the physics of fields—distributed systems involving expressions $\phi(t, x)$ into which t and x enter *conjointly* as independent variables. But it renders untenable (except as an informative approximation) the Newtonian dynamics of particles, since the latter theory involves expressions $x(t)$ into which God’s time—now abandoned—enters as a *solitary* independent variable. It is, in this light, not surprising to discover that Einstein, in Part II of his initial paper,¹⁰ found Maxwellian electrodynamics—the father of all field theories—to be it *already relativistic*, but that he was obliged to undertake a relativistic reconstruction of particle dynamics. It was from the latter effort that $E = mc^2$ emerged.

Tacking now in a different direction. . .

We inherit the “inertial observer” concept from Galileo (for whom Copernicus set the stage), by way of Newton. It is the business of Newton’s 1st Law to identify *who is qualified* to write $F = m\ddot{x}$, to do Newtonian physics. This is the test:¹¹ Turn off all force machines, launch a test particle, and see whether its motion, with respect to your Cartesian frame, can be described

$$\frac{d^2}{dt^2}x(t) = 0$$

i.e., whether you see the particle to move “uniformly and rectilinearly.” If so, you are an “inertial” observer—a potential physicist—and if not, you aren’t, and should think about majoring in some other field. By simple exercise in the calculus we discover that *if we are inertial, then so also is every observer—and only those—whom we see to be gliding by uniformly and rectilinearly*. We are led thus to the image of an *infinite population of inertial observers, in all possible states of uniform/rectilinear (and rotation-free) relative motion*.

Relativity implicitly abandons Newtonian dynamics, but at the same time retains the inertial observer concept—subject, however, to this qualification:

¹¹ I allow myself, for the purposes at hand, to ignore certain subtleties having mainly to do with rotational aspects of the matter, else with the circumstances which serve to define the extent of the “local neighborhood.”

it discards all observers O whose speed with respect to O exceeds c . The surprising fact is that this can be accomplished self-consistently, and in a way which promotes no observer to a distinguished place. The wonderousness of this development becomes more vividly apparent when one assumes spacetime to be three (or four) dimensional, as in the following figure:

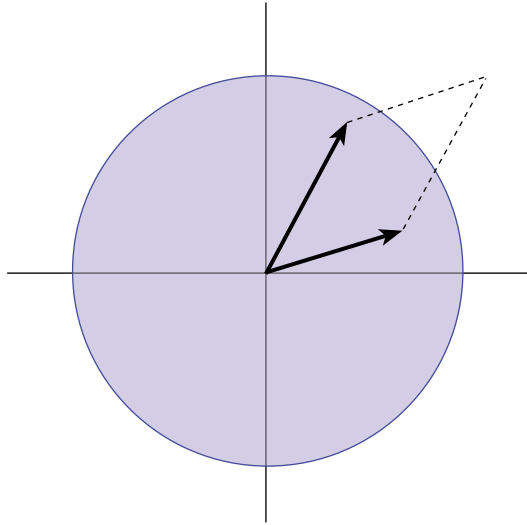


FIGURE 14: The “ c -ball” as it would appear if spacetime were three dimensional. It lives in velocity space (or β space), and shares the 2-dimensionality of space. Relativity excludes relative velocity vectors unless they lie internal to the ball of radius c . In Galilean physics one writes $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ to describe the result of successive (or “composed”) transformation, and such sums may venture into the forbidden exterior of the c -ball. In relativity, the modified “velocity addition formula” (21) serves neatly to prevent this from happening.

It is the special design of the “relativistic velocity addition formula”

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \equiv \beta(\beta_1, \beta_2) \quad (21)$$

which is here at work: clearly

$$\beta(0.8, 0.8) = 0.9756 \quad (\text{not } 1.6)$$

$$\beta(1.0, 0.8) = 1.0 \quad (\text{not } 1.8)$$

$$\beta(1.0, 1.0) = 1.0 \quad (\text{not } 2.0)$$

though $\beta(0.008, 0.008) = 0.015999 \sim 0.016000$. The function $\beta(\beta_1, \beta_2)$ is plotted in the final pair of figures.

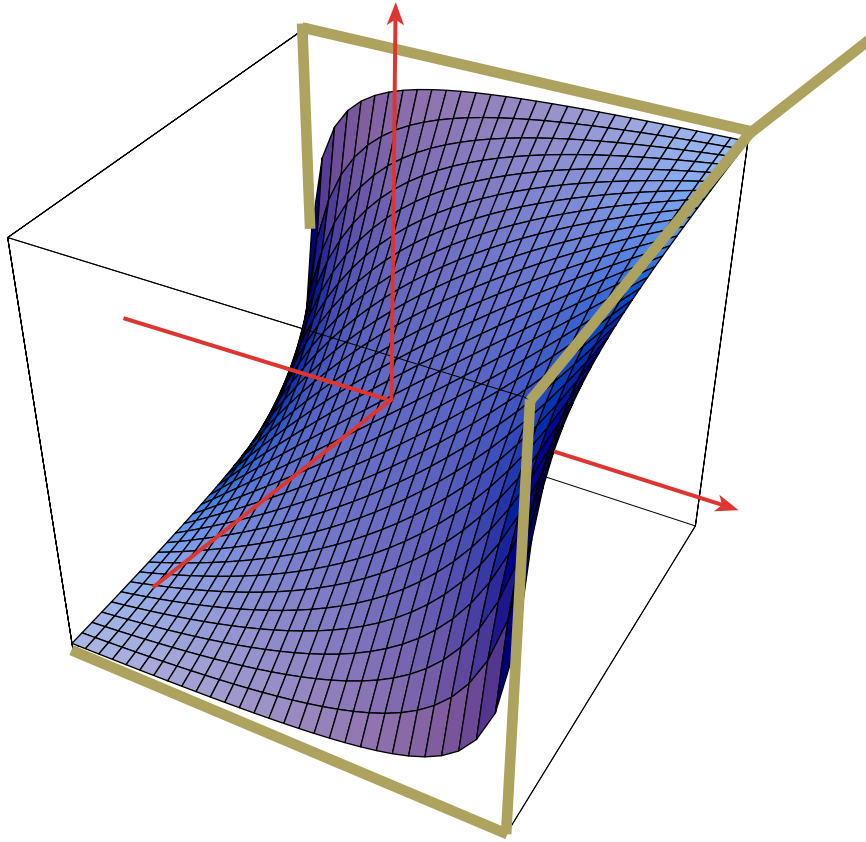


FIGURE 15: *Plot of the function $\beta(\beta_1, \beta_2)$. In the vicinity of the origin the “ β -surface” can be replaced by its tangent plane, and one in that approximation recovers the Galilean velocity addition formula $\beta = \beta_1 + \beta_2$. The wire frame has been introduced to suggest that the β -surface very nearly resembles the soap film which such a cubic skeleton would support.*

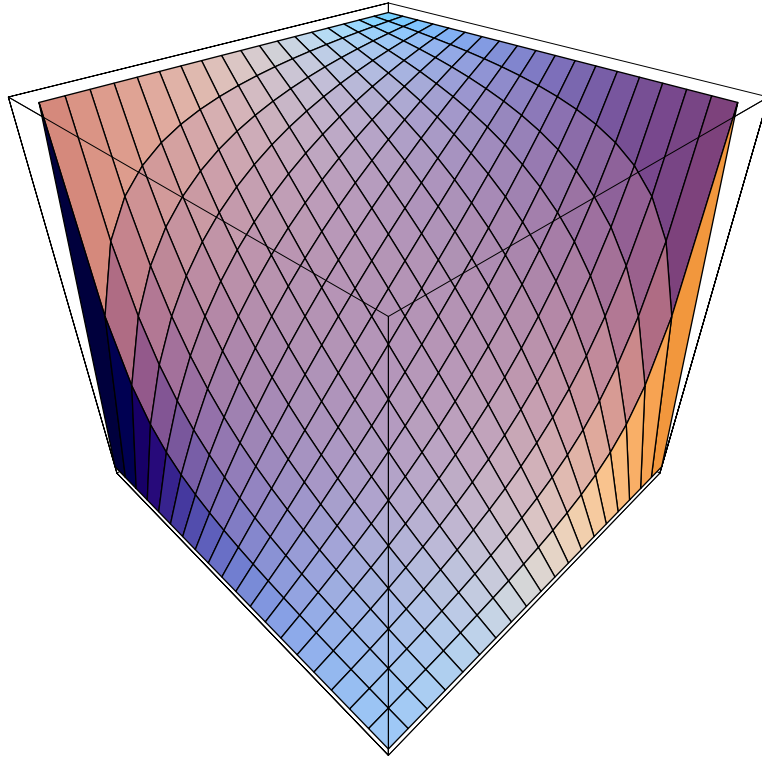


FIGURE 16: The β -surface viewed from the symmetrically disposed “neutral corner” $(1, 1, -1)$. The wire frame, from this vantage point, looks like a hexagon, with vertices alternating up/down/up/down/up/down. If one writes $\beta_3 \equiv -\beta(\beta_1, \beta_2)$ then (21) becomes

$$\beta_1 + \beta_2 + \beta_3 + \beta_1\beta_2\beta_3 = 0$$

—the symmetry of which conforms to the symmetry of the figure.

One must be careful not to conflate these two statements

- it is impossible for the speed, with respect to O , of an observer O to exceed c
- it is impossible for the speed, with respect to O , of a mass point m to exceed c

—ultimately related though they are. The former is central to Special Relativity its ownself, while the latter issues from relativistic *mechanics*, which is a more

pliable construct. That it is possible in principle to honor the former restriction while abandoning the latter was first appreciated by physicists in the 1960’s. For an account of the anticipated properties of what Arnold Sommerfeld (already in 1904) called “überlichtgeschwindigkeitsteilchen” and what are today usually called “tachyons”—particles that travel faster than light, see S. Tanaka, “Theory of matter with super light velocity,” *Prog. of Theo. Phys.* **24**, 171 (1960); O. Bilaniuk, V. Deshpande & E. C. G. Sudershan, “Meta’ relativity,” *AJP* **30**, 718 (1962) or G. Feinberg, “Possibility of faster-than-light particles,” *Phys. Rev.* **159**, 1089 (1967). Many of those properties are quite bizarre; for example, and for reasons evident in Figure 11, a particle (tachyon) which journeys $\circ \rightarrow \bullet$ to a point outside the lightcone centered upon \circ will appear to some observers to travel *backwards in time!*

And to almost everyone’s surprise, recent experiments¹² appear to have shown a certain subtle kind of faster-than-light data transmission to be a quantum mechanical fact of Nature; what appeared to Einstein, Boris Podolsky and Nathan Rosen¹³ in 1935 to be patently absurd evidently seems to God to be not so absurd.

Einstein’s youthful question (like again his question of 38 years later) opened the door to a mansion of many marvelous rooms. Many of those are now busy places. In others the windows have yet to be thrown open to the sunlight. And some, I suspect, remain to be explored for the first time.

¹² A. Aspect, P. Grangier & G. Roger, “Experimental tests of realistic local theories via Bell’s theorem,” *Phys. Rev. Letters* **47**, 460 (1981); “Experimental realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*: a new violation of Bell’s inequalities,” *Phys. Rev. Letters* **49**, 91 (1982) by those same authors; W. Tittel, J. Brendal, H. Zbinden & N. Gisin, “Violation of Bell inequalities by photons more than 10km apart,” *Phys. Rev. Letters* **81**, 3563 (1998). For an account of the intellectual soil from which that work derives, see Chapter 10: Einstein-Podolsky-Rosen Experiments in J. S. Bell, *Speakable and Unsayable in Quantum Mechanics: Collected Papers on Quantum Philosophy* (1987)

¹³ “Can quantum mechanical description of physical reality be considered complete?” *Phys. Rev.* **47**, 777 (1934). This brief classic—it runs to scarcely four pages—is reprinted at p. 138 in J. A. Wheeler & W. H. Zurek (editors), *Quantum Theory and Measurement* (1983).